Problem 5096. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}}\geq\frac{3}{2}$$

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By weighted AM-GM inequality we have

$$\begin{aligned}&\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}}\geq\\ \geq&\frac{a}{b+\frac{1}{4}a+\frac{3}{4}b}+\frac{b}{c+\frac{1}{4}b+\frac{3}{4}c}+\frac{c}{a+\frac{1}{4}c+\frac{3}{4}a}=\\ =&\frac{4a}{a+7b}+\frac{4b}{b+7c}+\frac{4c}{c+7a}\end{aligned}$$

so it suffices to prove that

$$\frac{a}{a+7b}+\frac{b}{b+7c}+\frac{c}{c+7a}\geq\frac{3}{8}$$

This inequality is equivalent to

$$\frac{7\left(13a^2b+13b^2c+13ac^2+35ab^2+35a^2c+35bc^2-144abc\right)}{8(a+7b)(b+7c)(c+7a)}\geq 0$$

which is true. Indeed according AM-GM inequality we obtain

$$13a^2b+13b^2c+13ac^2\geq 13\cdot 3\cdot\sqrt[3]{a^3b^3c^3}=39abc$$

$$35ab^2+35a^2c+35bc^2\geq 35\cdot 3\cdot\sqrt[3]{a^3b^3c^3}=105abc$$

and, summing these inequalities, we get

$$13a^2b+35ab^2+35a^2c-144abc+13b^2c+13ac^2+35bc^2\geq 144abc$$

This ends the proof. Clearly, equality occurs for $a=b=c$. $\square$